Large Order Behavior of Quasiclassical Euclidean Gravity in Minisuperspace Models

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Abstract

We demonstrate in two minisuperspace models that a perturbation expansion of quasiclassical Euclidean gravity has a factorial dependence on the order of the term at large orders. This behavior indicates that the expansion is an asymptotic series which is suggestive of an effective field theory. The series may or may not be Borel summable depending on the classical solution expanded around. We assume that only the positive action classical solution contributes to path integrals. We close with some speculative discussion on possible implications of the asymptotic nature of the expansion.

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1 Introduction

The starting point of our analysis is the Euclidean action of gravity [1]:

$$\hat{I} = -\frac{1}{16\pi G}\int_{\mathcal{M}}(R-2\Lambda)\sqrt{g}d^4x. \tag{1}$$

This action comes from the Euclidean continuation of the Hilbert action. We include the cosmological constant term but we will be interested mainly in Λ near zero. The Euclidean partition function is then:

$$Z = \int D[g]\exp(-\hat{I}[g]). \tag{2}$$

There are many problems with this definition of gravity as a fundamental theory, (see [2]), which we are not going to discuss in this letter. Rather, we want to look at gravity as an effective field theory described by an effective Lagrangian, L_{eff}, which, by definition, contains operators of arbitrary dimensionality. This view of gravity as an effective field theory of an unknown high energy fundamental theory was advocated for a many years by Weinberg[3], see also the recent paper[5]. As was argued recently [6], the corresponding expansion in general is not a convergent but rather an asymptotic series with factorially growing coefficients. Let us note that this remark about the factorial dependence of the series is an absolutely irrelevant issue for the analysis of low energy phenomena. We have nothing new to say about these issues. However, if we wish to discuss Plank scale phenomena, we need to know the behavior of the whole series when distant terms in the series might be important. Only in this case does the analysis of the large order terms in the expansion have some physical meaning.

The first application of this idea to problems of inflation was discussed recently [7]. It was argued that the effective potential which is obtained by resumming the series provides a natural realization of the inflationary scenario. Therefore, it would be worthwhile to prove the main assumption of the paper [7] about factorial growth of coefficients in the effective lagrangian. The **main goal** of the present letter is to explicitly demonstrate the factorial dependence of quasiclasssical Euclidean gravity in two minisuperspace models.

The series may or may not be Borel summable depending on which classical solution the path integral is expanded about. We assume following [4]

that only classical solutions with positive action are relevant. It should be noted, however, that the factorial dependence is independent of these details. We discuss some applications in the conclusion.

2 A Simple Minisuperspace Model

Assume a metric of the form:

$$ds^2 = dt^2 + a(t)^2 d\Omega^2. \tag{3}$$

The Euclidean action in this case is:

$$\hat{I} = -\frac{3\pi}{4G}\int_0^T dt\sqrt{a^6}\left[a^{-2}(1-\dot{a}^2) - \ddot{a}a^{-1} - \frac{\Lambda}{3}\right]. \tag{4}$$

We consider this as an analytical functional of a complex scale factor 'a(t)'. We stress this point since it is usual in general relativity to consider only real scale factors, in which case the action would be strictly negative after choosing the positive sign volume form. Treating 'a(t)' as a complex number, however, we can write the action in the form:

$$\hat{I} = -\frac{3\pi}{4G}\int_0^T dt\left[a(1-\dot{a}^2) - \ddot{a}a^2 - \frac{\Lambda}{3}a^3\right], \tag{5}$$

which can have arbitrary phase. This agrees with the action in [8] after integration by parts. Both forms lead to the same results in the leading classical term. At higher orders, surface terms which are neglected in [8] become important. Both forms of the action have vanishing variation for solutions to:

$$1 - \dot{a}^2 - 2a\ddot{a} - \Lambda a^2 = 0, \tag{6}$$

and resulting classical solutions are $a_{cl} = \pm\sqrt{\frac{3}{\Lambda}}\sin(\sqrt{\frac{\Lambda}{3}}t)$. The positive solution is the usual solution and the only solution considered in [8]. It gives a negative action. The second solution produces a positive action.

In either case, restricting ourselves to a compact Euclidean geometry in the classical limit by selecting the final time $T = \pi\sqrt{3/\Lambda}$, we obtain a universe that expands from zero size at t=0 to a maximum and recontracts to a point at t=T. The manifold described by this metric is S^4 and the spacetime it describes is called Euclidean De Sitter space.

The surface terms from the previously mentioned integration by parts vanish because a_{cl} vanishes at the endpoints of the integration. Note also that the manifold has no boundary which means that a boundary integration over extrinsic curvature (K), which we have neglected, cannot contribute.

At this point we can substitute the value of the classical action into the path integral (2) to obtain in the classical approximation:

$$Z \propto \exp\left(\pm\frac{3\pi}{\Lambda G}\right) \tag{7}$$

where the signs agree with the sign of the classical solution. The existence of a classical solution with finite Euclidean action means that this solution can be used to calculate the large order behavior of the perturbation theory as was suggested many years ago [10],[9].

Notice that as $\Lambda \to 0$ the first solution blows up while the second is perfectly well behaved. Klebanov et al. [8] used the first solution. Since we are looking for a theory that has meaning for vanishing cosmological constant, we choose to do a quasiclassical approximation about the other classical solution, $a_{cl} = -\sqrt{\frac{3}{\Lambda}}\sin(\sqrt{\frac{\Lambda}{3}}t)$. This is the same solution one would get by performing a Hawking rotation on 'a(t)' in the action integral, solving and then continuing back to real 'a(t)'(see [8]). Thus, we obtain the same result without using the Hawking prescription.

With only the classical solution we can now derive the factorial dependence of the higher order terms in the quasiclassical expansion. We would like to consider the path integral as a function of a small coupling constant $g = \Lambda/3$ and find its expansion in 'g':

$$Z(g) = \int D[a]\,\exp\left(\frac{3\pi}{4G}\int_0^T dt\left[a(1+\dot{a}^2) - ga^3\right]\right) \equiv \sum_{K=0}^{\infty} Z_K g^K. \tag{8}$$

For large values of the classical action, this functional integral over complex 'a(t)' is dominated by the value of the integrand at the corresponding classical solution. Using the nontrivial value of the classical action and techniques from [10],[9] (see also review [11]) we obtain:

$$Z_K \propto \int_{-\infty}^{0}\frac{dg}{g^{K+1}}\,\exp\left(\frac{\pi}{gG}\right) = (-\frac{G}{\pi})^K\Gamma(K) \approx (-\frac{G}{\pi})^K K! \tag{9}$$

for large K when we expand about the negative action classical solution. Therefore, this expansion has the promised factorial dependence.

Expanding about the positive action classical solution requires a modification of the formulas from [11] but gives the following result for large K:

$$Z_K \propto \int_0^\infty \frac{dg}{g^{K+1}} \exp\left(-\frac{\pi}{gG}\right) = (\frac{G}{\pi})^K \Gamma(K) \approx (\frac{G}{\pi})^K K!. \tag{10}$$

Notice that both cases have the factorial dependence characteristic of an asymptotic series. The only difference between the two cases is that the negative action solution has alternating sign whereas the positive action solution does not. The alternating sign could make the series Borel summable, but we believe the classical solution with positive action is the physically favorable solution[4]. In principle all saddle points may contribute to the functional integral and which saddle points actually do contribute is determined by the definition of the functional integral.

The quasiclassical expansion about this solution is:

$$\hat{I}[a_{cl}+\delta a] \approx \hat{I}_{cl}+\frac{1}{2}\int_0^T dt\delta a(t)\left[-2\Lambda a_{cl} - 2\ddot{a}_{cl} - 2\dot{a}_{cl}\frac{d}{dt} - 2a_{cl}\frac{d^2}{dt^2}\right]\delta a(t) \tag{11}$$

The effect of quadratic fluctuations can be determined by expanding the perturbation $\delta a(t)$ in an orthogonal set of eigenfunctions of the quadratic operator. Without actually solving the problem, we know that the eigenvalue spectrum is bounded below since the operator is Sturm—Liouville. The same differential equation would apply to the expansion about the negative action classical solution except with the opposite sign on the operator. Therefore its eigenvalue spectrum is bounded above and the negative action solution is highly unstable. We do not know how to handle this problem, which is another reason we do not consider the negative action solution.

It can be easily shown that the quadratic operator about the positive action solution has a zero eigenvalue and at least one negative eigenvalue. The instability of the positive action classical solution due to a finite number of negative eigenmodes is much preferable to the extremely unstable negative action classical solution with an infinite number of negative eigenmodes. The dependence of the quasiclassical prefactor can be evaluated if the eigenvalue spectrum is known in the first case. The problem is undefined in the latter case.

We propose to avoid divergences and regularization prescriptions by only considering contributions to the partition function from the classical solution

with positive action. This is a prescription that we cannot completely justify, but is supported by arguments due to Marolf [4]. His arguments do not completely carry over to our case, because the Hamiltonian is not positive definite, but they lend credence to our assumption.

There is reason to believe that the apparent singularity of the Euclidean partition function arises as a result of an incorrect continuation of the partition function from Minkowski space. It has been shown [13],[14] that the correct Euclidean path integral resembles the conformally rotated naive Euclidean path integral, except for an extra Faddeev-Popov determinant factor. The path integral defined in this way is nonsingular and well defined. Our prescription leads to a well defined path integral and the classical solution agrees with the conformally rotated path integral.

Finally, although we have discussed the first nonvanishing variations about the classical solution in this minisuperspace model, we have not calculated this prefactor. The general expansion about the De Sitter solution was studied in [15] and the prefactor for our case could be determined using their results, but this is not required for our purposes.

3 ϕ^4 Theory

A second, more general, minisuperspace model arises from the metric[8]:

$$ds^2 = \left(\frac{4\pi G}{3}\phi^2\delta_{ij}\right) dx^i \wedge dx^j. \tag{12}$$

The Euclidean action in this case is:

$$\hat{I} = -\int d^4x[\frac{1}{2}(\partial\phi)^2 + \frac{1}{4}g\phi^4] \tag{13}$$

where $g = -8\pi G\Lambda/9$. The path integral is:

$$Z = \int D[\phi]\ \exp\left(\int d^4x[\frac{1}{2}(\partial\phi)^2 + \frac{1}{4}g\phi^4]\right). \tag{14}$$

This is similar to the path integral for Euclidean ϕ^4 theory except for the unconventional sign. The large order behavior of ϕ^4 theory is known [10],[11] to have factorial dependence on the order with alternating sign terms. This

occurs because the theory is expanded about a complex saddle point[2]. The path integral in our case will retain the factorial dependence but, as in the previous model, the alternating sign is absent. The results are:

$$Z_K \propto \left(\frac{8\pi G\Lambda}{9} I(\phi_{cl})\right)^{-K} K! \tag{15}$$

where ϕ_{cl} is the instanton:

$$\phi_{cl}(r) = \sqrt{\frac{9}{\pi G\Lambda}}\ \frac{\lambda}{[1+\lambda^2(r-r_0)^2]} \tag{16}$$

with action $I(\phi_{cl}) = \frac{3\pi}{\Lambda G}$, corresponding to Euclidean De Sitter space. This model intersects the previous model at its classical solutions.

The quasiclassical prefactor can be determined from the results of [15],[10].

4 Conclusion and Wild Speculations

The main result of this paper is the factorial dependence on order of higher order terms in the quasiclassical expansion of Euclidean gravity. We have shown this in two minisuperspace models using only the value of the classical action. This dependence is characteristic of an asymptotic series. This is just the type of behavior we would expect of a low energy effective field theory. The factorial growth of expansion coefficients was assumed in [7] to deal with problems of inflation.

The expansion may or may not be Borel summable depending on the classical solution one expands about. We chose to expand about the positive action solution. We assumed following [4] that only the positive action solution contributes to the Euclidean path integral and expanded about that solution, but this assumption does not affect the general statement above.

In the first minisuperspace model we argued that at least one negative eigenmode exists, but left the calculation of quasiclassical prefactor for the future. The second minisuperspace model is very similar to ϕ^4 theory for which the first quasiclassical prefactor can be obtained from [10].

[2]There are no instantons in a real scalar field theory, but these calculations are based on the analytical continuation of such a field theory and so may have complex instantons.

The nonrenormalizable nature of gravity — the most profound field theoretic feature — does not show up at the classical level in the minisuperspace models considered in this paper (10,15). Indeed, any ultraviolet (UV) divergences will appear in our approach only at the level of calculation of the one-loop quantum determinant. The corresponding divergences can be absorbed in the standard way by redefining the original parameters of the ϕ^4 theory. It can be done explicitly due to the renormalizability of the obtained ϕ^4 theory (13). The procedure is well defined, but seems to involve an exchange of limit between large cut-off and large order. We make a standard field theory assumption that such an exchange is justified and does not affect the main result of factorial growth. Note that a similar assumption is not required for the first minisuperspace model which is reduced to a quantum mechanical system (5) rather than a field theory.

In general, however, because of the dimensionality of the coupling constant G, one loop calculations generate operators proportional not to the original action R, but higher order terms such as R^2, $R_{\mu\nu}R^{\mu\nu}$. This is an explicit manifestation of the non-renormalizability of gravity. However, when gravity is treated as an effective field theory it can be renormalized at any given order by absorbing the divergences into renormalized values of the coefficients in the most general effective Lagrangian. In this respect the theory resembles the effective chiral Lagrangian approach where one can show that coefficients also exhibit factorial growth[6]. More importantly, this growth is not affected by the UV divergences at least at the one loop level.

Indeed, in the semiclassical approximation the UV divergences appear only at the level of the calculation of the quantum determinant. As usual, this determinant is divergent but factorized from the classical part. Therefore, the quantum part is independent of the $k!$ behavior, which originates from the classical action. Technically, the factor related to ultraviolet regularization at one-loop level will appear in front of $k!$:

$$Z_k \sim \frac{1}{\epsilon} k!, \quad \epsilon = d - 4. \tag{17}$$

One can absorb this divergence at one loop level by redefining any counterterm from the action. After that, formula (17) gives a finite result for arbitrary high terms and explicitly demonstrates the expected $k!$ behavior.

One could stop here if we accept the Hawking viewpoint [1] that the dominant contribution to the functional integral can be represented as a sum

of background and one-loop terms **only**! This point is motivated by an idea that all classical solutions (i.e. all metrics) with all possible topologies are dense in some sense in the space of all metrics. If this is the case, one could then hope (see [1]) to pick out some finite number of solutions which give the dominant contribution to the path integral (spacetime foam picture).

This prescription for handling the nonrenormalizability of quantum gravity has survived for many years. Nevertheless, we believe it might be interesting to discuss another possibility for treating this problem which should be considered, at the moment, as wild speculation at best.

The basic idea, as before, is the observation that the perturbation series (8) is an asymptotic one. Therefore, we can use some integral representation formula for this series. For illustration purposes we assume that the series is Borel summable and we use a Borel representation for it [3]:

$$Z(g) = \sum_{K=0}^{\infty} Z_K g^K \sim \int_0^{\infty} \frac{f(t)dt}{t(t+g)} \exp(-\frac{1}{t}), \tag{18}$$

where a function $f(t)$ is defined by moments

$$Z_K \sim (-)^K \int_0^{\infty} \frac{f(t)dt}{t^{K+1}} \exp(-\frac{1}{t}) \sim (-)^K K! \tag{19}$$

and should be mild enough to preserve the main asymptotical behavior $\sim K!$. This would be the end of the story if we were to discuss a quantum mechanical problem. However, we wish to discuss a nonrenormalizable field theory where, at the one-loop level, an UV divergence will appear in front of the $K!$ factor, as in (17). However, the most singular behavior in gravity which

[3] We already mentioned that the series (8) which represents our system most likely is not Borel summable. We believe, however, that the Borel non-summability of an expansion does not signal an inconsistency or ambiguity of the theory. The Borel prescription is just one of many summation methods and need not be applicable everywhere. Thus, some prescription, based on the physical considerations, should be given in order to evaluate an integral like this. Some new physics usually accompanies such a phenomenon, but we do not go into details here. Rather, we would like to mention the non-Borel summable example of the principal chiral field theory at large N[16]. In this case, the explicit solution is known. The coefficients grow factorially with the order and the series is not Borel summable. Nevertheless, the physical observables are perfectly well defined and the exact result can be recovered by a special prescription which uses a non-trivial procedure of analytic continuation.

could occur in front of G^K is not the one-loop divergence, $\frac{1}{\epsilon}$, we discussed previously, but rather the K-th loop divergence proportional to $\frac{1}{\epsilon^K}$. Therefore, in general, we expect the following structure for the K-th loop term in quantum gravity:

$$Z_K \sim \frac{K!}{\epsilon^K} c_0^{(K)} (1 + c_1^{(K)} \epsilon + c_2^{(K)} \epsilon^2 + ...), \qquad \epsilon = d - 4. \tag{20}$$

Now, if we use the Borel prescription (18) to sum up this series, then we would get a result of zero for this series:

$$Z(g) = \sum_{K=0}^{\infty} Z_K \frac{G^K}{\epsilon^K} \sim \int_0^{\infty} \frac{\exp(-\frac{1}{t}) f(t) dt}{t(t + \frac{G}{\epsilon}(1 + 0(\epsilon))} \sim \epsilon \to 0, \tag{21}$$

in spite of the fact that each term on the left hand side diverges in the limit $\epsilon = d - 4 \to 0$ and irrespective of the precise behavior of the coefficients, $c_0^{(K)}$, which presumably can be modeled by a function $f(t)$. The finite terms at each level apparently have very different analytical structure (see [5]) and should be treated separately from UV divergent terms (21).

We are not pretending to have made a reliable analysis of UV divergences in gravity in this letter. Rather, we wanted to point out that an asymptotic nature of the expansion might provide a natural way to handle the problem of UV divergences in gravity. As we mentioned, there is a possibility that each term in the series is divergent, but the series itself is a well defined function. At least, we cannot rule out this possibility from the very beginning and we believe it deserves further investigation.

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